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On the Solution of the Diophantine Equation $A^3 + B^3 + C^3 + D^3 = 0$

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This paper presents a complete and symmetric solution of the diophantine equation of the title in terms of four parameters α, β, γ, δ, subject to $\alpha + \beta + \gamma + \delta = 0$.

By solving the diophantine equation

$$A^3 + B^3 + C^3 + D^3 = 0, \tag{1}$$

we mean finding a system of functions $F_i(\alpha, \beta,...)$, $(i = 1, 2, 3, 4)$, such that

(i) these functions are rational and integral in the terms of the parameters α, β,...,

(ii) they have no common polynomial factors, and

(iii) they satisfy Eq. (1) identically, that is

$$\sum_{i=1}^{4} F_i^3(\alpha, \beta,...) \equiv 0. \tag{2}$$

The solution is called *complete* if for any four integers A, B, C, and D, satisfying (1) there can be found integral parameters α, β,... and a rational proportionality factor ρ, such that

$$\begin{aligned} A &= \rho F_1(\alpha, \beta,...), \\ B &= \rho F_2(\alpha, \beta,...), \\ C &= \rho F_3(\alpha, \beta,...), \\ D &= \rho F_4(\alpha, \beta,...). \end{aligned} \tag{3}$$

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The solution is called *symmetric* if the four functions $F_1, \ldots, F_4$ have a similar structure, or more precisely if, by applying a certain permutation group to the order of parameters $\alpha, \beta, \ldots$ in the individual functions $F_1, \ldots, F_4$, these transform into one another.

Starting with Vieta Eq. (1) or its equivalent forms have been considered by several mathematicians. The first thorough discussion appeared in 1770 in Euler's book "Algebra." Apart from the trivial cases, Euler's solution is complete. The formulas are in general quoted in a simplified form deduced by Binet (see [1]) in 1841. Hurwitz ([3], or see [2]) and Ögmundsson [4] gave new derivations to the Euler-type formulas, while the solutions of Dickson, [1], and Ramanujan (see [2]) are essentially different. The former is entirely complete and the latter is incomplete. As for the question of symmetry, it is remarkable that no known solution is symmetric in the sense defined above.

To arrive at a complete system of solutions which is also symmetric, I propose the following. Since

$$A^3 + B^3 = (A + B)(A + \epsilon B)(A + \epsilon^2 B)$$

and

$$C^3 + D^3 = (C + D)(C + \epsilon D)(C + \epsilon^2 D),$$

where $\epsilon = \exp(2\pi i/3)$, let

$$\begin{aligned} A + \epsilon B &= (\alpha + \epsilon\beta)(\mu + \epsilon\nu), \\ A + \epsilon^2 B &= (\alpha + \epsilon^2\beta)(\mu + \epsilon^2\nu), \\ C + \epsilon D &= (\gamma + \epsilon\delta)(\mu + \epsilon\nu), \\ C + \epsilon^2 D &= (\gamma + \epsilon^2\delta)(\mu + \epsilon^2\nu). \end{aligned} \tag{4}$$

Then Eq. (1) is satisfied if

$$\begin{aligned} A + B &= \lambda(\gamma + \epsilon\delta)(\gamma + \epsilon^2\delta) = \lambda(\gamma^2 - \gamma\delta + \delta) \\ \text{and} \quad C + D &= -\lambda(\alpha + \epsilon\beta)(\alpha + \epsilon^2\beta) = -\lambda(\alpha^2 - \alpha\beta + \beta^2) \end{aligned} \tag{5}$$

for some rational λ. Now, (4) is equivalent to

$$\begin{aligned} A &= \alpha\mu - \beta\nu, \\ B &= \beta(\mu - \nu) + \alpha\nu, \\ C &= \gamma\mu - \delta\nu, \\ D &= \delta(\mu - \nu) + \gamma\nu, \end{aligned} \tag{6}$$

and hence, by substituting these expressions for A, B, C, and D in (5), we have

$$\mu(\alpha + \beta) + \nu(\alpha - 2\beta) = \lambda(\gamma^2 - \gamma\delta + \delta^2)$$

and

$$\mu(\gamma + \delta) + \nu(\gamma - 2\delta) = -\lambda(\alpha^2 - \alpha\beta + \beta^2). \tag{7}$$

These equations can be solved to obtain μ and ν as functions of the parameters α, β, γ, δ, and λ. These functions may in turn be substituted in (4) to obtain solutions for Eq. (1). These solutions, however, contain an excessive number of parameters, for, as is obvious, we need only three independent parameters for a complete system of solutions $F_1, \ldots, F_4$. Hence we can postulate two arbitrary relations between the parameters and the proportionality factor ρ, and in order to bring about symmetry, we set

$$\lambda = \rho(\alpha + \beta) = -\rho(\gamma + \delta). \tag{8}$$

Thus

$$\alpha + \beta + \gamma + \delta = 0, \tag{9}$$

and so, by adding the two equations at (7), we eliminate μ:

$$\begin{aligned}\nu(\alpha - 2\beta + \gamma - 2\delta) &\\ = 3\nu(\alpha + \gamma) &= \rho(\alpha + \beta)(\gamma^2 - \gamma\delta + \delta^2 - \alpha^2 + \alpha\beta - \beta^2)\\ = 3\rho(\alpha + \beta)(\alpha\beta - \gamma\delta) &= 3\rho(\alpha + \beta)(\alpha + \gamma)(\beta + \gamma),\end{aligned}$$

that is,

$$\nu = \rho(\alpha + \beta)(\beta + \gamma) = \rho(\alpha\gamma - \beta\delta). \tag{10}$$

For μ we obtain the more complicated formula,

$$\mu = \rho[\alpha^2 - \alpha\beta + \beta^2 - (\alpha + \delta)(\gamma - 2\delta)],$$

or

$$\mu = \rho[\gamma^2 - \gamma\delta + \delta^2 - (\beta + \gamma)(\alpha - 2\beta)]. \tag{11}$$

Substituting (10) and (11) in (6) and also considering (9) and (3), we obtain the following expressions for the functions $F_1, \ldots, F_4$:

$$\begin{aligned}F_1(\alpha, \beta, \gamma, \delta) &= 2\alpha^3 + \alpha^2\beta + 2\alpha\beta^2 + 4\alpha^2\delta + 2\alpha\beta\delta + \beta^2\delta + 3\alpha\delta^2,\\ F_2(\alpha, \beta, \gamma, \delta) &= F_1(\beta, \alpha, \delta, \gamma),\\ F_3(\alpha, \beta, \gamma, \delta) &= F_1(\gamma, \delta, \alpha, \beta),\end{aligned} \tag{12}$$

and

$$F_4(\alpha, \beta, \gamma, \delta) = F_1(\delta, \gamma, \beta, \alpha).$$

Hence we have the Abelian noncyclic group of order 4 (Klein's group), which is acting on the parameters α, β, γ, and δ.

That the system of solutions given by (12) is complete is evident from the following. Suppose we have found four numbers A, B, C, and D, satisfying (1) in a nontrivial way. Then $A + B + C + D \neq 0$, and the equation

$$\frac{\mu}{\nu} = \frac{2A + 2C - B - D}{A + B + C + D}, \tag{13}$$

which is obtained from (6) and (9) by eliminating the parameters, determines the ratio of μ and ν. Next we solve system (6) for the parameters:

$$\begin{aligned}
\alpha &= \frac{A(\mu - \nu) + B\nu}{\mu^2 - \mu\nu + \nu^2}, \\
\beta &= \frac{B\mu - A\nu}{\mu^2 - \mu\nu + \nu^2}, \\
\gamma &= \frac{C(\mu - \nu) + D\nu}{\mu^2 - \mu\nu + \nu^2}, \\
\delta &= \frac{D\mu - C\nu}{\mu^2 - \mu\nu + \nu^2},
\end{aligned} \tag{14}$$

In the trivial case, i.e., when $A + B + C + D = 0$, we let

$$\alpha : \beta : \gamma : \delta = A : B : C : D.$$

Thus, in both cases, once A, B, C, and D are given, the values of α, β, γ, and δ are determined up to a constant factor. The fact that the numbers A, B, C, and D satisfy (1), then guarantees that (5) or the equivalent systems, (7) or (10) and (11), also hold true, and consequently the numbers A, B, C, and D are representable in the form defined by (3) and (12).

Further Problems. By permuting the numbers A, B, C, and D in (13) and (14), we obtain three different ratios of the auxiliary parameters μ and ν, each ratio leading to two different quadruples of parameters α, β, γ, and δ. The question of how these ratios and sets of parameters are related to each other and to the Dickson and Ramanujan solutions will be discussed in a second paper of the author on the subject.

References

1. L. E. Dickson, "Introduction to the Theory of Numbers," Section 36, Chap. 4.
2. G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," Section 13.7, Oxford, 1971.
3. A. Hurwitz, *Math. Werke* **2** (1933), 469–470.
4. V. Ögmundsson, A complete integral solution of $x^3 + y^3 = u^3 + v^3$, *Nordisk Mat. Tidskr.* **13** (1965), 88–90.